Pricing European and Barrier Options in the Fractional Black-Scholes Market

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Abstract: The aim of this paper is to obtain the valuation formulas for European and barrier options if the underlying of the option contract is supposed to be driven by a fractional Brownian motion with Hurst parameter greater than 0.5. The paper is build upon the framework developed in Necula (2007) for the valuation of derivative products in the fractional Black-Scholes market. We also obtain a reflection principle for the fractional Brownian motion.

Keywords: fractional Brownian motion, fractional Black-Scholes market, the reflection principle for the fractional Brownian motion, mathematical finance, European option, barrier option

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1. INTRODUCTION

The fractional Brownian motion (fBm) with Hurst parameter H ($0 < H < 1$) is a self-similar continuous Gaussian process with long memory. The self-similarity and long-range dependence properties make the fractional Brownian motion a suitable tool in mathematical finance. It was proved in Hu and Oksendal (2003) that the fractional Black-Schools market, based on the stochastic integral developed by Duncan, Hu and Pasik-Duncan (2000), has no arbitrage. Hu and Oksendal (2003) introduced the concept of quasi-conditional expectation and quasi-martingales. Necula (2002) and Necula (2007) developed a framework for the valuation of contingent claims in the fractional Black-Scholes market using the risk-neutral methodology based on the quasi-conditional expectation.

The aim of this paper is to obtain the valuation formulas for European and barrier options if the underlying of the option contract is supposed to be driven by a fractional Brownian motion with Hurst parameter greater than 0.5.

Hu and Oksendal (2003) obtained a formula for the price of a European call option at $t = 0$. The purpose of this article is to extend the formula for every $t \in [0,T]$. After an earlier version of this article (Necula, 2002) we become aware of a similar result of Bender (2004) who used fractional BSDE.

This paper is organized as follows: in the second section we prove the valuation formula for the European call option in the fractional Black-Scholes market. In the third section we obtain the price of barrier options when the interest rate is zero as well as a reflection principle for the fractional Brownian motion.

2. EUROPEAN OPTIONS VALUATION IN THE FRACTIONAL BLACK-SCHOLES MODEL

THEOREM 2.1. (fractional Black-Scholes formula)

The price at every $t \in [0,T]$ of an European call option with strike price K and maturity T is given by

$$C(t,S(t)) = S(t)N(d_1) - Ke^{-r\cdot(T-t)}N(d_2)$$

where $$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$ and

$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

and $N(\cdot)$ is the cumulative probability of the standard normal distribution.

Proof:

Using the fractional risk-neutral evaluation (theorem 3.1 in Necula, 2007) we have that

$$C(t, S(t)) = \widetilde{E}_t\left[e^{-r\cdot(T-t)} \max((S(T)-K),0)\right]$$

$$= \widetilde{E}_t\left[e^{-r\cdot(T-t)} S(T)\, \mathbb{1}_{\{S(T)>K\}}\right] - Ke^{-r\cdot(T-t)} \widetilde{E}_t\left[\mathbb{1}_{\{S(T)>K\}}\right]$$

$$= e^{-r\cdot(T-t)} \widetilde{E}_t\left[S(T)\, \mathbb{1}_{\{S(T)>K\}}\right] - Ke^{-r\cdot(T-t)} \widetilde{E}_t\left[\mathbb{1}_{\{S(T)>K\}}\right]$$

But if we denote by

$$d_2^* = \frac{\ln\left(K/S\right) - rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma}$$

we get

$$\widetilde{E}_t\left[\mathbb{1}_{\{S(T)>K\}}\right] = \widetilde{E}_t\left[\mathbb{1}_{\{x>d_2^*\}}\left(B_H(T)\right)\right]$$

$$= \int_{d_2^*}^{\infty} \frac{1}{\sqrt{2\pi\left(T^{2H} - t^{2H}\right)}} \exp\left(-\frac{\left(x - B_H(t)\right)^2}{2\left(T^{2H} - t^{2H}\right)}\right) dx$$

$$= \int_{\frac{d_2^* - B_H(t)}{\sqrt{T^{2H} - t^{2H}}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= \int_{-\infty}^{\frac{B_H(t) - d_2^*}{\sqrt{T^{2H} - t^{2H}}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$= N(d_2)$$

Consider the process

$$B_H^*(t) = B_H(t) - \sigma t^{2H}, \quad 0 \le t \le T$$

Theorem 3.18 in Hu and Oksendal, (2003) assures us that there is a measure P^* such that $B_H^*(t)$ is a fractional Brownian motion under P^*.

We will denote

$$Z(t) = \exp\left(\sigma B_H(t) - \frac{\sigma^2}{2} t^{2H}\right)$$

Using Theorem 2.4 in Necula (2007) we have that

$$\begin{aligned}
\widetilde{E}_t\left[S(T)\,\mathbb{1}_{\{S(T) > K\}}\right] &= e^{rT}\widetilde{E}_t\left[Z(T)\,\mathbb{1}_{\{x > d_2^*\}}(B_H(T))\right] \\
&= e^{rT} Z(t) \widetilde{E}_t^*\left[\mathbb{1}_{\{x > d_2^*\}}(B_H(T))\right] \\
&= e^{rT} Z(t) \widetilde{E}_t^*\left[\mathbb{1}_{\{S(T) > K\}}\right]
\end{aligned}$$

But

$$\begin{aligned}
\ln(S(T)) &= \ln S + rT - \frac{\sigma^2}{2} T^{2H} + \sigma B_H(T) \\
&= \ln S + rT + \frac{\sigma^2}{2} T^{2H} + \sigma B_H^*(T)
\end{aligned}$$

If we denote

$$d_1^* = \frac{\ln\left(K/S\right) - rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma}$$

we get

$$\widetilde{E}_t^*\left[\mathbb{1}_{\{S(T) > K\}}\right] = \widetilde{E}_t^*\left[\mathbb{1}_{\{x > d_1^*\}}(B_H^*(T))\right]$$

$$= \int_{d_1^*}^{\infty} \frac{1}{\sqrt{2\pi\left(T^{2H} - t^{2H}\right)}} \exp\left(- \frac{\left(x - B_H^*(t)\right)^2}{2\left(T^{2H} - t^{2H}\right)} \right) dx$$

$$= \int_{\frac{d_1^* - B_H^*(t)}{\sqrt{T^{2H} - t^{2H}}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{z^2}{2} \right) dz$$

$$= \int_{-\infty}^{\frac{B_H^*(t) - d_1^*}{\sqrt{T^{2H} - t^{2H}}}} \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{z^2}{2} \right) dz$$

$$= N\left(d_1\right)$$

So

$$\widetilde{E}_t\left[S(T)\, \mathbb{1}_{\{S(T) > K\}} \right] = e^{rT} Z(t) N\left(d_1\right)$$

$$= e^{rT} e^{-rt} S(t) N\left(d_1\right)$$

The result follows immediately.

q.e.d.

THEOREM 2.2 (The Greeks)
The Greeks of the European call are given by:

$$\Delta = \frac{\partial C}{\partial S} = N\left(d_1\right)$$

$$\nabla = \frac{\partial C}{\partial K} = -e^{-r(T-t)} N\left(d_2\right)$$

$$\vartheta = \frac{\partial C}{\partial \sigma} = Sf\left(d_1\right)\sqrt{T^{2H} - t^{2H}}$$

$$\rho = \frac{\partial C}{\partial r} = (T - t) K e^{-r(T-t)} N\left(d_2\right)$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{1}{S\sigma\sqrt{T^{2H} - t^{2H}}} f\left(d_1\right)$$

$$\Theta = \frac{\partial C}{\partial t} = -rKe^{-r(T-t)} N\left(d_2\right) - Ht^{2H-1} \frac{S\sigma}{\sqrt{T^{2H} - t^{2H}}} f\left(d_1\right)$$

where

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

<u>Proof:</u>

We will first derive a general formula. Let y be one of the influence factors.

We have

$$\frac{\partial C}{\partial y}=\frac{\partial S}{\partial y}N(d_1)+S\frac{\partial N(d_1)}{\partial y}-\frac{\partial\left(Ke^{-r\cdot(T-t)}\right)}{\partial y}N(d_2)-Ke^{-r\cdot(T-t)}\frac{\partial N(d_2)}{\partial y}$$

But

$$\frac{\partial N(d_2)}{\partial y}=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_2{}^2}{2}\right)\frac{\partial d_2}{\partial y}=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(d_1-\sigma\sqrt{T^{2H}-t^{2H}}\right)^2}{2}\right)\frac{\partial d_2}{\partial y}$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_1{}^2}{2}\right)\exp\left(d_1\sigma\sqrt{T^{2H}-t^{2H}}\right)\exp\left(-\frac{\sigma^2\left(T^{2H}-t^{2H}\right)}{2}\right)\frac{\partial d_2}{\partial y}$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_1{}^2}{2}\right)\exp\left(\ln\left(\frac{S(t)}{K}\right)+r(T-t)\right)\frac{\partial d_2}{\partial y}$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_1{}^2}{2}\right)\frac{S(t)}{K}\exp(r(T-t))\frac{\partial d_2}{\partial y}$$

It follows that:

$$\frac{\partial C}{\partial y}=\frac{\partial S}{\partial y}N(d_1)-\frac{\partial\left(Ke^{-r\cdot(T-t)}\right)}{\partial y}N(d_2)+Sf(d_1)\frac{\partial\left(\sigma\sqrt{T^{2H}-t^{2H}}\right)}{\partial y}\tag{3.3}$$

Substituting in 3.3 we get the Greeks.

q.e.d.

3. BARRIER OPTIONS VALUATION IN THE FRACTIONAL BLACK-SCHOLES MODEL

First let consider the contingent claims:

- Binary call and put with strike K:

$$BC = \mathbb{1}_{\{S(T) > K\}}, \quad BP = \mathbb{1}_{\{S(T) < K\}}$$

- Gap call and put with strike K:

$$GC = S(T)\,\mathbb{1}_{\{S(T) > K\}}, \quad GP = S(T)\,\mathbb{1}_{\{S(T) < K\}}$$

Using Theorem 2.4 and Theorem 3.1 in Necula (2007) we have, with the notations in Theorem 2.1, the following lemma:

LEMMA 3.1 *We have that:*

$$BC(t, S(t)) = e^{-r(T-t)} N(d_2)\ , \ BP(t, S(t)) = e^{-r(T-t)} N(-d_2)$$

$$GC(t, S(t)) = S(t) N(d_1),\ GP(t, S(t)) = S(t) N(-d_1)$$

We will make the following notations:

$$\tau_L := \inf\{t \mid S(t) = L\}$$
$$m^S(T) := \inf_{0<t<T} S(t)$$
$$M^S(T) := \sup_{0<t<T} S(t)$$

THEOREM 3.2. *Consider that* $r = 0$.

1. *If* $K > L$, $S > L$ *and* $t < \tau_L$ *then*

$$\widetilde{E}_t\left[\mathbb{1}_{\{S(T) > K, m^S(T) > L\}}\right] = N(a_1) - \frac{S(t)}{L} N(a_2)$$

where $a_1 = \dfrac{\ln\left(\dfrac{S(t)}{K}\right) - \dfrac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$

and $$a_2 = \frac{\ln\left(\frac{H^2}{LS(t)}\right) - \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

2. *If* $S > L$ *and* $t < \tau_L$ *then*

$$\widetilde{E}_t\left[\mathbb{1}_{\{\tau_L > T\}}\right] = \widetilde{E}_t\left[\mathbb{1}_{\{m^S(T) > L\}}\right] = N(b_1) - \frac{S(t)}{L}N(b_2)$$

where $$b_1 = \frac{\ln\left(\frac{S(t)}{L}\right) - \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

and $$b_2 = \frac{\ln\left(\frac{L}{S(t)}\right) - \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

3. *If* $K < L$, $S < L$ *and* $t < \tau_L$ *then*

$$\widetilde{E}_t\left[\mathbb{1}_{\{S(T) < K, M^S(T) < L\}}\right] = N(-a_1) - \frac{S(t)}{L}N(-a_2)$$

4. *If* $S < L$ *and* $t < \tau_L$ *then*

$$\widetilde{E}_t\left[\mathbb{1}_{\{\tau_H > T\}}\right] = \widetilde{E}_t\left[\mathbb{1}_{\{M^S(T) < L\}}\right] = N(-b_1) - \frac{S(t)}{L}N(-b_2)$$

Proof:

1. Consider a down-and-out binary call (DOBC) with strike price K, barrier L and maturity T. The payoff of this option is $\mathbb{1}_{\{S(T) > K, m^S(T) > L\}}$. The price of this contingent claim $DOBC(t)$ is nonzero if $t < \tau_L$ and zero if $t > \tau_L$.

Consider now a portfolio that consists in a long position of one binary call with strike K and maturity T and a short position of $\frac{1}{L}$ gap puts with strike price $\frac{L^2}{K}$ and maturity T. It can be seen from Lemma 4.1 that if $t = \tau_L$ (i.e. $S(t) = L$) the price of this portfolio is zero. So if the barrier is hit previous to the maturity T the value of this portfolio is equal to that of the option. If the barrier is not hit the portfolio and the option will have the same payoff at maturity since the gap put is out of the money ($S(T) > L, K > L \Rightarrow S(T) > \frac{L^2}{K}$).

Since the fractional Black-Scholes does not have arbitrage and the down-and-out binary call and the portfolio have the same payoff, their value will be the same for $t < \tau_L \wedge T$. So

$$DOBC(t) = BC(t) - \frac{1}{L} GP(t)$$

2. Consider a contingent claim that pays one unit if the stock price does not hit the barrier L before T (down-and-out bond). The payoff of this contingent claim is $\mathbb{1}_{\{m^S(T) > L\}}$. As in the previous case we look for a portfolio that has the same value as the contingent claim at $\tau_L \wedge T$, and as a consequence of the no-arbitrage property of the fractional Black-Scholes they will have the same value for every $t < \tau_L \wedge T$.

   In this case we can chose a portfolio consisting in a long position of one binary call with strike L and maturity T and a short position of $\frac{1}{L}$ gap puts with strike price L and maturity T .

3. In this case we consider an up-and-out binary put and a portfolio that consists in a long position of one binary put with strike K and maturity T and a short position of $\frac{1}{L}$ gap calls with strike price $\frac{L^2}{K}$ and maturity T .
4. In this case we consider an up-and-out bond and a portfolio that consists in a long position of one binary put with strike L and maturity T and a short position of $\frac{1}{L}$ gap calls with strike price L and maturity T .

q.e.d.

THEOREM 3.3. Consider that $r = 0$.

1. If $K > L$, $S > L$ and $t < \tau_L$ then

$$\tilde{E}_t\left[S(T)\mathbb{1}_{\{S(T) > K, m^S(T) > L\}} \right] = S(t)N(c_1) - L\,N(-c_2)$$

where $c_1 = \dfrac{\ln\left(\dfrac{S(t)}{K}\right) + \dfrac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$

and $c_2 = \dfrac{\ln\left(\dfrac{KS(t)}{L^2}\right) - \dfrac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$

2. *If* $K < L$, $S < L$ *and* $t < \tau_L$ *then*

$$\widetilde{E}_t\left[S(T)\mathbb{1}_{\{S(T) < K, M^S(T) < L\}}\right] = S(t)N(-c_1) - L\,N(c_2)$$

Proof:

1. Consider a down-and-out call with strike price K, barrier L and maturity T. The payoff of this option is $(S(T) - K)\mathbb{1}_{\{S(T) > K, m^S(T) > L\}}$. The value of this option at $t < \tau_L \wedge T$ is given by:

$$\widetilde{E}_t\left[S(T)\mathbb{1}_{\{S(T) < K, M^S(T) < L\}}\right] - K\,\widetilde{E}_t\left[\mathbb{1}_{\{S(T) < K, M^S(T) < L\}}\right]$$

One can see that this contingent claim has the same payoff as a portfolio that consists in a long position of one call with strike K and maturity T and a short position of $\frac{K}{L}$ puts with strike price $\frac{L^2}{K}$ and maturity T. Using Theorem 4.2 the value of this portfolio at $t < \tau_L \wedge T$ is:

$$S(t)N(c_1) - L\,N(-c_2) - K\left(N(a_1) - \frac{S(t)}{L}N(a_2)\right)$$

2. In this case we consider an up-and-out put and a portfolio that consists in a long position of one put with strike K and maturity T and a short position of $\frac{K}{L}$ calls with strike price $\frac{L^2}{K}$ and maturity T.

q.e.d.

The reflection principle of the Brownian motion gives the common distribution of the Brownian motion and its minimum (or maximum). The next corollary gives a similar result for the fractional Brownian motion.

Consider $B_H^*(t) = -\frac{\sigma}{2}t^{2H} + B_H(t)$. It is known that there is a is a probability measures P^* such that $B_H^*(t)$ is a fBm under P^*.

COROLLARY 3.4.

1. If $y \le 0$, $x > y$ and $t < \tau_y$ then

$$\widetilde{E}_t\left[\mathbb{1}_{\{B_H^*(T) > x, m^{B_H^*}(T) > y\}} \right] = N\left(\frac{B_H^*(t) - x + \frac{\sigma}{2}\left(T^{2H} - t^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}} \right) -$$

$$- \exp\left\{\sigma\left(B_H^*(t) - y\right)\right\} N\left(\frac{2y - x - B_H^*(t) + \frac{\sigma}{2}\left(T^{2H} - t^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}} \right)$$

2. If $y \ge 0$, $x < y$ and $t < \tau_y$ then

$$\widetilde{E}_t\left[\mathbb{1}_{\{B_H^*(T) < x, M^{B_H^*}(T) < y\}} \right] = N\left(\frac{x - B_H^*(t) - \frac{\sigma}{2}\left(T^{2H} - t^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}} \right) -$$

$$- \exp\left\{\sigma\left(B_H^*(t) - y\right)\right\} N\left(\frac{x - 2y + B_H^*(t) - \frac{\sigma}{2}\left(T^{2H} - t^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}} \right)$$

The shortcoming of the result is that the quasi-conditional expectation it is under P, not under the probability measure P^* ($B_H^*(t)$ is a fBm under P^*, not under P).

4. CONCLUSION

In this paper we obtained the valuation formulas for European and barrier options if the underlying of the option contract is supposed to be driven by a fractional Brownian motion with Hurst parameter greater than 0.5.

The fractional Black-Scholes price of a European call option no longer depends only on $T-t$ as in the classical model (Black and Scholes, 1973). A reason may be the fact that the fractional Brownian motion has long memory. The price of an option at a moment $t \in [0,T]$ will depend on the stock price $S(t)$, but despite the classical Black-Scholes model, will take into consideration the evolution of the stock price in the period $[0,t]$. This influence is reflected in the fractional Black-Scholes formula by the Hurst parameter H.

Consider three moments $t_1 \le t_2 \le t \le T$ and two options with maturity T one of them written on t_1 and the other one on t_2. In the classical Black-Scholes model the prices of the two options at the moment t were equal. In the fractional Black-Scholes model the prices of the two options at the moment t are no longer equal. Due to the long memory property, the price of the first option is also influenced by the evolution of the stock price in the period $[t_1, t_2]$.

We obtained the price down-and-out call barrier option in the particular case that the risk free interest rate is zero. A more interesting result would be a formula for the price of barrier options in the case in which the interest rate is not zero. But the extension to the case of non-zero drift seems very quite difficult.

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