A Framework for Derivative Pricing in the Fractional Black-Scholes Market

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Abstract: The aim of this paper is to develop a framework for evaluating derivatives if the underlying of the derivative contract is supposed to be driven by a fractional Brownian motion with Hurst parameter greater than 0.5. For this purpose we first prove some results regarding the quasi-conditional expectation, especially the behavior to a Girsanov transform. We obtain the risk-neutral valuation formula and the fundamental evaluation equation in the case of the fractional Black-Scholes market.

Keywords: fractional Brownian motion, fractional Black-Scholes market, quasi-conditional expectation, mathematical finance, contingent claim

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1. INTRODUCTION

The fractional Brownian motion (fBm) with Hurst parameter H ($0 < H < 1$) is the continuous Gaussian process $\{B_H(t), t \in \mathbb{R}\}$, $B_H(t) = 0$ with mean $E[B_H(t)] = 0$ and whose covariance is given by:

$$C_H(t,s) = E[B_H(t)B_H(t)] = \frac{1}{2}\left\{ |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right\}$$

If $H = \frac{1}{2}$ then $B_H(t)$ coincides with the standard Brownian motion $B(t)$.

The fractional Brownian motion is a self-similar process meaning that for any $\alpha > 0$ $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$.

The parameter H determines the sign of the covariance of the future and past increments. This covariance is positive when $H > \frac{1}{2}$, zero when $H = \frac{1}{2}$ (i.e. the classical Brownian motion) and negative when $H < \frac{1}{2}$.

Another property of the fractional Brownian motion is that for $H > \frac{1}{2}$ it has long-range dependence in the sense that if we put

$$r(n) = Cov(B_H(1), B_H(n+1) - B_H(n))$$

then

$$\sum_{n=1}^{\infty} r(n) = \infty$$

The self-similarity and long-range dependence properties make the fractional Brownian motion a suitable tool in different applications like mathematical finance. Since for $H \neq \frac{1}{2}$ the fractional Brownian motion is neither a Markov process, nor a semimartingale (see for exemple Rogers, 1997), we cannot use the usual stochastic calculus to analyse it. Worse still, after a pathwise integration theory for fractional Brownian motion was developed by Lin (1995) and Decreusefond and Ustunel (1999), it was proven by Rogers (1997) that the market mathematical models driven by $B_H(t)$ could have arbitrage. The fractional Brownian motion was no longer considered fit for mathematical modeling in finance. However after the development by Duncan, Hu and Pasik-Duncan (2000) and Hu and Oksendal (2003) of a new kind of integral based on the Wick product called the fractional Ito integral, it was proved in Hu and Oksendal (2003) that the corresponding Ito type fractional Black-Schools market has no arbitrage. Equivalent definitions of the fractional Ito integral were introduced by Alos, Mazet and Nualart (2000), Perez-Abreau and Tudor (2002) and Bender (2002). Hu and Oksendal (2003) introduced the concept of quasi-conditional expectation and quasi-martingales. In the same paper a formula for the price of a European option at $t = 0$ is derived.

The aim of this paper is to build a framework for evaluating derivatives if the underlying of the derivative contract is supposed to be driven by a fractional Brownian motion with Hurst parameter greater than 0.5.

This paper is organized as follows: in the first section we prove some results regarding the quasi-conditional expectation, especially the behaviour to a Girsanov transform. In the third section we apply these results to obtain the risk-neutral valuation formula and the fundamental evaluation equation in the case of the fractional Black-Scholes market. The final section concludes.

2. SOME RESULTS REGARDING THE QUASI-CONDITIONAL EXPECTATION

Let $(\Omega, \mathcal{K}, P)$ a probability field such that $B_H(t,\omega)$ is a fractional Brownian motion with respect to P, $\mathcal{F}_t^H := \mathcal{B}(B_H(s), s \le t)$ and $\widetilde{E}_t[\ ]$ the *quasi-conditional expectation* with respect to $\mathcal{F}_t^H$ (definition 4.9 in Hu and Oksendal, 2003).

THEOREM 2.1. *For every* $0 < t < T$ *and* $\lambda \in \mathbb{C}$ *we have that*

$$\widetilde{E}_t\left[e^{\lambda B_H(T)}\right] = e^{\lambda B_H(t) + \frac{\lambda^2}{2}\left(T^{2H} - t^{2H}\right)}$$

Proof:

Consider the fractional differential equation:

$$dX(t) = \lambda X(t) dB_H(t), \quad X(0) = 1$$

The solution of this equation is (Hu and Oksendal, 2003):

$$X(t) = \exp\left(\lambda B_H(t) - \frac{1}{2}\lambda^2 t^{2H}\right)$$

Since

$$X(t)=\int_0^t \lambda X(s)dB_H(s)$$

we know that $X(t)$ is a quasi-martingale (Hu and Oksendal, 2003). So it follows that:

$$\widetilde{E}_t[X(T)]=X(t)$$

or

$$\widetilde{E}_t\left[e^{\lambda B_H(T)}\right]=e^{\lambda B_H(t)+\frac{\lambda^2}{2}\left(T^{2H}-t^{2H}\right)}$$

q.e.d.

THEOREM 2.2. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a function such that* $E[f(B_H(T))]<\infty$. *Then for every* $t\le T$

$$\widetilde{E}_t[f(B_H(T))]=\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\left(T^{2H}-t^{2H}\right)}}\exp\left(-\frac{(x-B_H(t))^2}{2\left(T^{2H}-t^{2H}\right)}\right)f(x)dx\ .$$

Proof:

Let $\hat{f}$ be the Fourier transform of f:

$$\hat{f}(\xi)=\int_{\mathbb{R}}e^{-ix\xi}f(x)dx$$

Then f is the inverse Fourier transform of $\hat{f}$:

$$f(x)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ix\xi}\hat{f}(\xi)d\xi$$

We have that:

$$f(B_H(T))=\frac{1}{2\pi}\int_{\mathbb{R}}e^{iB_H(T)\xi}\hat{f}(\xi)d\xi$$

It follows that:

$$\widetilde{E}_t[f(B_H(T))]=\widetilde{E}_t\left[\frac{1}{2\pi}\int_{\mathbb{R}}e^{i\xi B_H(T)}\hat{f}(\xi)d\xi\right]$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \widetilde{E}_t \left[e^{i\xi B_H(T)} \right] \hat{f}(\xi) d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi B_H(t) - \frac{\xi^2}{2}\left(T^{2H} - t^{2H}\right)} \hat{f}(\xi) d\xi$$

$$= h(B_H(t))$$

where h is the inverse Fourier transform of the product between $e^{-\frac{\xi^2}{2}\left(T^{2H} - t^{2H}\right)}$ and $\hat{f}(\xi)$.

But the first function is the Fourier transform of

$$n_{t,T}(x) = \frac{1}{\sqrt{2\pi\left(T^{2H} - t^{2H}\right)}} \exp\left(- \frac{x^2}{2\left(T^{2H} - t^{2H}\right)} \right)$$

Using the fact that the Fourier transform of a convolution is the product of the Fourier transform of the two functions it follows that

$$h(B_H(t)) = \int_{\mathbb{R}} n_{t,T}(B_H(t) - y) f(y) dy$$

q.e.d.

COROLLARY 2.3. *Let* $A \in \mathcal{B}(\mathbb{R})$. *Then*

$$\widetilde{E}_t\left[\mathbb{1}_A(B_H(T))\right] = \int_A \frac{1}{\sqrt{2\pi\left(T^{2H} - t^{2H}\right)}} \exp\left(- \frac{(x - B_H(t))^2}{2\left(T^{2H} - t^{2H}\right)} \right) dx$$

Let $\theta \in \mathbb{R}$. Consider the process

$$B_H^*(t) = B_H(t) + \theta\, t^{2H} = B_H(t) + \int_0^t 2H\theta\, \tau^{2H-1} d\tau, \quad 0 \le t \le T$$

Theorem 3.18 in Hu and Oksendal (2003) assures us that there is a measure P^* such that $B_H^*(t)$ is a fractional Brownian motion under P^*.

We will denote $\widetilde{E}_t^*[\cdot]$ the quasi-conditional expectation with respect to P^*.

Consider

$$Z(t) := \exp\left(-\theta B_H(t) - \frac{\theta^2}{2} t^{2H}\right)$$

THEOREM 2.4. *Let* f *be a function such that* $E\left[f\left(B_H(T)\right)\right] < \infty$. *Then for every* $t \le T$

$$\widetilde{E}_t^*\left[f\left(B_H(T)\right)\right] = \frac{1}{Z(t)} \widetilde{E}_t\left[f\left(B_H(T)\right) Z(T)\right]$$

Proof:

Again we will denote by $\hat{f}$ the Fourier transform of f.

We have

$$\widetilde{E}_t\left[f\left(B_H(T)\right)Z(T)\right] = \widetilde{E}_t\left[\frac{1}{2\pi}\int_R e^{(i\xi-\theta)B_H(T)-\frac{\theta^2}{2}T^{2H}} \hat{f}(\xi)d\xi\right]$$

$$= \frac{1}{2\pi} e^{-\frac{\theta^2}{2}T^{2H}} \int_R \widetilde{E}_t\left[e^{(i\xi-\theta)B_H(T)}\right]\hat{f}(\xi)d\xi$$

$$= \frac{1}{2\pi} e^{-\frac{\theta^2}{2}T^{2H}} \int_R e^{(i\xi-\theta)B_H(t)+\left(-\frac{\xi^2}{2}-i\xi\theta+\frac{\theta^2}{2}\right)\left(T^{2H}-t^{2H}\right)} \hat{f}(\xi)d\xi$$

$$= Z(t)\frac{1}{2\pi}\int_R e^{i\xi B_H(t)+\left(-\frac{\xi^2}{2}-i\xi\theta\right)\left(T^{2H}-t^{2H}\right)} \hat{f}(\xi)d\xi$$

On the other hand

$$\widetilde{E}_t^*\left[f\left(B_H(T)\right)\right] = \widetilde{E}_t^*\left[\frac{1}{2\pi}\int_R e^{i\xi\left(B_H^*(T)-\theta T^{2H}\right)} \hat{f}(\xi)d\xi\right]$$

$$= \frac{1}{2\pi}\int_R \widetilde{E}_t^*\left[e^{i\xi B_H^*(T)}\right]e^{-i\xi\theta T^{2H}} \hat{f}(\xi)d\xi$$

$$= \frac{1}{2\pi}\int_R e^{i\xi B_H^*(t)-\frac{\xi^2}{2}\left(T^{2H}-t^{2H}\right)}e^{-i\xi\theta T^{2H}} \hat{f}(\xi)d\xi$$

$$= \frac{1}{2\pi}\int_R e^{i\xi\left(B_H(t)+\theta t^{2H}\right)-\frac{\xi^2}{2}\left(T^{2H}-t^{2H}\right)}e^{-i\xi\theta T^{2H}} \hat{f}(\xi)d\xi$$

$$= \frac{1}{2\pi} \int_R e^{i\xi B_H(t) + \left(-\frac{\xi^2}{2} - i\xi\theta\right)\left(T^{2H} - t^{2H}\right)} \hat{f}(\xi) d\xi$$

The result follows immediately.

q.e.d.

3. RISK NEUTRAL VALUATION IN THE FRACTIONAL BLACK-SCHOLES MARKET

Consider the *fractional Black-Scholes market* consisting of two investment possibilities:

1. a money market account:

$$dM(t) = rM(t)dt, \quad M(0) = 1, \quad 0 \le t \le T$$

where r represent the constant riskless interest rate.

2. a stock whose price satisfies the equation:

$$dS(t) = \mu\, S(t)dt + \sigma\, S(t)d\overline{B}_H(t), \quad S(0) = S > 0,\ 0 \le t \le T$$

where $\mu, \sigma \neq 0$ are constants and $\overline{B}_H(t)$ is a fractional Brownian motion with respect to the market measure.

In Hu and Oksendal (2003) it was shown that this market does not have arbitrage and is complete, the same properties as the classical Black-Scholes model based on the Brownian motion.

Under the risk-neutral measure (denoted P) we have that:

$$dS(t) = rS(t)dt + \sigma S(t)dB_H(t), \quad S(0) = S > 0,\ 0 \le t \le T$$

where $B_H(t)$ is a fractional Brownian motion with respect to P.

The solution of this stochastic differential equation is (Hu and Oksendal, 2003):

$$S(t) = S \exp\left(\sigma\, B_H(t) + r\, t - \frac{1}{2}\sigma^2 t^{2H}\right)$$

We will denote by $\widetilde{E}_t[\cdot]$ the quasi-conditional expectation with respect to the risk-neutral measure.

THEOREM 3.1. (*fractional risk-neutral evaluation*)

The price at every $t \in [0,T]$ *of a bounded* $\mathcal{F}_T^H$ *- measurable contingent claim* $F \in L^2(P)$ *is given by*

$$F(t) = e^{-r(T-t)} \widetilde{E}_t [F]$$

Proof:

Since the market is complete there is a replicating portfolio of the claim $(m(t), s(t))$ whose value is:

$$F(t) = m(t)M(t) + s(t)S(t)$$

and

$$F(T) = F$$

We have that

$$\begin{aligned} dF(t) &= m(t)dM(t) + s(t)dS(t) \\ &= rF(t)dt + \sigma\, s(t)S(t)dB_H(t) \end{aligned}$$

By multiplying with e^{-rt} and integrating it follows that

$$e^{-rt}F(t) = F(0) + \int_0^t e^{-r\tau}\sigma\, s(\tau)S(\tau)dB_H(\tau),\ 0 \le t \le T \qquad (3.1)$$

By the fractional Clark-Ocone theorem (theorem 4.15 in Hu and Oksendal, 2003) we have that

$$e^{-rT}F = E\left[e^{-rT}F\right] + e^{-rT}\int_0^T \widetilde{E}_\tau [D_\tau F]dB_H(\tau)$$

where $D_\tau F$ is the Malliavin derivative of F (definition 4.3 in Hu and Oksendal, 2003).

From the completeness of the market we get

$$\widetilde{E}_\tau [D_\tau F] = e^{r(T-\tau)}\sigma\, s(\tau)S(\tau),\ 0 \le \tau \le T$$

So we have that

$$e^{-rT}F = E\left[e^{-rT}F\right] + \int_0^T e^{-r\tau}\sigma\, s(\tau)S(\tau)dB_H(\tau)$$

It follows that

$$\widetilde{E}_t\left[e^{-rT}F\right] = E\left[e^{-rT}F\right] + \widetilde{E}_t\left[\int_0^T e^{-r\tau}\sigma\, s(\tau)S(\tau)dB_H(\tau)\right]$$

Using the fact that $\int_0^t e^{-r\tau}\sigma\, s(\tau)S(\tau)dB_H(\tau)$ is a quasi-martingale we have:

$$\widetilde{E}_t\left[e^{-rT}F\right]=E\left[e^{-rT}F\right]+\int_0^t e^{-r\tau}\sigma\, s(\tau)S(\tau)dB_H(\tau) \tag{3.2}$$

From 3.1 and 3.2 we have that

$$F(t)=e^{-r(T-t)}\widetilde{E}_t\left[F\right]$$

q.e.d.

THEOREM 3.2 (*fractional fundamental evaluation equation*)

The price of a derivative on the stock price with a bounded payoff $f\left(S(T)\right)$ *is given by* $F\left(t,S(t)\right)$*, where* $F\left(t,S\right)$ *is the solution of the PDE:*

$$\frac{\partial F}{\partial t}+H\sigma^2t^{2H-1}S^2\frac{\partial^2F}{\partial S^2}+rS\frac{\partial F}{\partial S}-rF=0$$
$$F(T,S)=f\left(S\right)$$

Proof:

From Theorem 3.1 and Theorem 2.2 it follows that the price at a moment t of the derivative with payoff $f\left(S(T)\right)$ is a function of t and $S(t)$.

As in the classical Black-Scholes model (Black and Scholes, 1973) we consider a portfolio that contains a derivative and $-\Delta$ stock.

The value of this portfolio is

$$\Pi(t)=F\left(t,S(t)\right)-\Delta S(t)$$

Under the market measure $\overline{P}$ using the fractional Ito formula (theorem 4.5 in Duncan, Hu and Pasik-Duncan, 2000) and the fact that

$$\begin{aligned}D_\tau S(\tau)&=S(\tau)D_u\left(\sigma B_H(\tau)+\mu\tau-\frac{\sigma^2}{2}\tau^{2H}\right)\\&=S(\tau)D_u\left(\sigma B_H(\tau)\right)\\&=\sigma S(\tau)\mathbb{1}_{[0,\tau]}(u)\end{aligned}$$

and

$$D_\tau^\phi S(\tau)=\sigma S(\tau)\int_0^\tau\phi(\tau,u)du=\sigma HS(\tau)\tau^{2H-1}$$

we get that

$$d\Pi(t)=dF\left(t,S(t)\right)-\Delta dS(t)$$

$$= \left(\frac{\partial F}{\partial t} + H\sigma^2 t^{2H-1} S^2 \frac{\partial^2 F}{\partial S^2} + \mu\, S \frac{\partial F}{\partial S} - \Delta\mu\, S \right) dt + \left(\sigma S \frac{\partial F}{\partial S} - \Delta\sigma S \right) d\overline{B}_H(t)$$

We want this portfolio to be riskless. So

$$\Delta = \frac{\partial F}{\partial S} \qquad \text{and} \qquad d\Pi(t) = r\Pi(t)dt$$

It follows that the evaluation equation is given by:

$$\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + H\sigma^2 t^{2H-1} S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0$$

q.e.d.

4. CONCLUSION

In this paper it was developed a framework for evaluating derivatives if the underlying of the derivative contract is supposed to be driven by a fractional Brownian motion with Hurst parameter greater than 0.5.

We proved that in the fractional Black-Scholes market one can use the risk-neutral evaluation methodology but by using the quasi-conditional expectation. Thus, in this context, the price of a contingent claim is the quasi-conditional expectation of the present value of the future cash-flows generated by this financial product.

We also obtained for the fractional Black-Scholes market the fundamental evaluation equation of a contingent claim. As in the classical Black-Scholes model the fundamental equation does not depend on the expected return (μ), but on the riskless interest rate (r).

The risk-neutral evaluation methodology or the fundamental evaluation equation can be used to price a large class of derivatives in the context of fractional Black-Scholes market. For example, the price of an European call option with strike price K cab be obtained by solving the PDE in Theorem 3.3 with the boundary condition $F(T, S) = \max(S - K, 0)$.

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